PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A
MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

The Asymptotic Solution of Linear Second Order Differential Equations in a Domain Containing a Turning Point and a Regular Singularity

R. C. Thorne

Phil. Trans. R. Soc. Lond. A 1957 **249**, 585-596
doi: 10.1098/rsta.1957.0007

THE ASYMPTOTIC SOLUTION OF LINEAR SECOND ORDER DIFFERENTIAL EQUATIONS IN A DOMAIN CONTAINING A TURNING POINT AND A REGULAR SINGULARITY†

By R. C. THORNE

Trinity College, University of Cambridge, and The California Institute of Technology, Pasadena, California

(*Communicated by Sir Harold Jeffreys, F.R.S.—Received* 28 *August* 1956)

CONTENTS

Asymptotic solutions of the differential equation

$$d^2w/dz^2 = \{u^2z^{-2}(z_0-z)\,p_1(z)+z^{-2}q_1(z)\}\,w,$$

for large positive values of u are examined; $p_1(z)$ and $q_1(z)$ are regular functions of the complex variable z in a domain in which $p_1(z)$ does not vanish. The point $z=0$ is a regular singularity of the equation and a branch-cut extending from $z=0$ is taken through the point $z=z_0$ which is assumed to lie on the positive real z axis. Asymptotic expansions for the solutions of the equation, valid uniformly with respect to z in domains including $z=0$, $z=z_0\pm i0$, are derived in terms of Bessel functions of large order. Expansions given by previous theory are not valid at all these points. The theory can be applied to the Legendre functions.

1. Introduction and summary

Let u be a large positive parameter and let z be a complex variable lying in an open simply-connected domain $\mathbf{D}_z$ in which $p_1(z)$ and $q_1(z)$ are regular functions of z. Let z_0, a be real positive numbers and let $p_1(z)$ and $q_1(z)$ be real when z is real. In an earlier paper (Thorne 1957, hereafter referred to as I), the asymptotic expansions of the differential equation

$$\left.\begin{aligned} &\frac{d^2w}{dz^2}=\{u^2p(z)+q(z)\}\,w, \\ \text{where}\quad &p(z)=z^{-2}(z_0-z)\,p_1(z), \quad q(z)=z^{-2}q_1(z), \\ \text{and}\quad &p_1(z)=\frac{a^2}{z_0}+zO(1), \quad q_1(z)=b+zO(1) \quad \text{as} \quad z\to 0, \end{aligned}\right\} \tag{1·1}$$

were examined for $p_1(z)$, $q_1(z)$, a, z_0 satisfying the conditions above. Several different examples of this equation (1·1) were considered. The point $z=0$ is a regular singularity

† Based partly on research prepared under contract Nonr-220(11) between the U.S. Office of Naval Research and the California Institute of Technology. Reference no. NR 043-121.

of (1·1), and to ensure that the solutions of (1·1) should be single-valued, it is necessary to introduce into $\mathbf{D}_z$ a cut, $\mathbf{C}$ say, extending from $z = 0$.

This present work gives a further investigation for two of the cases considered in I. Following the classification introduced by Olver (1954*a*) these were designated case B 1*b* and case B 4. In case B 1*b* (I, § 3) the cut $\mathbf{C}$ is taken along the real positive axis through $z = z_0$ and the points $z_0 + i0$ and $z_0 - i0$ are written z_0^+ and z_0^-; we take $p_1(z)$ to be non-zero in $\mathbf{D}_z$. It was then proved that, providing (1·1) is modified so that $z^2 q_1(z) \to -\frac{1}{4}$ as $z \to 0$, there exist asymptotic expansions for solutions of (1·1) which are valid uniformly with respect to z on a Riemann surface in which $z = 0$, $z = z_0^+$ are interior points, but the expansions are not valid at $z = z_0^-$ and are not valid uniformly in the strip $\mathscr{R}z > z_0 - \delta$, $0 \geqslant \mathscr{I}z > -\delta$ $(\delta > 0)$, $|\arg z| \leqslant 2\pi$. In case B 4 (I, § 7) the function $p_1(z)$ is of the form

$$z^{-2}(z_0 - z)\, p_1(z) = z^{-2}(z_0^2 - z^2)\, p_2(z), \tag{1·2}$$

where $p_2(z)$ is an even function of z; in this paper we shall suppose that in case B 4 $q_1(z)$ is also an even function of z. In this case, $\mathbf{C}$ is taken along the real negative axis, and as in case B 1*b* there exist asymptotic expansions for solutions of (1·1) which are valid uniformly on a Riemann surface in which $z = 0$ and $z = z_0$ are interior points. The expansions are not valid at $z = -z_0$ and are not valid uniformly in the strip $\mathscr{R}z < -z_0 + \delta$, $|\mathscr{I}z| < \delta$ $(\delta > 0)$. We take, as mentioned above, both $p_2(z)$ and $q_1(z)$ to be even functions of z. By making the transformation $x = z^2$ we can then show that, under these circumstances, case B 4 is a particular example of case B 1*b*. For this reason we shall now confine our attention in this paper to case B 1*b*. The expansions of I for case B 1*b* and case B 4 were derived, as in all the cases considered in I, by an application of Olver's theory (1954*a*), and are in terms of Airy functions. In the introduction of I, a summary of Olver's theory was given together with a review of the other types of equation (1·1) discussed in I.

The point $z = z_0$, and any other point at which the coefficient of $u^2 w$ in (1·1) has a simple zero, is known as a turning point of (1·1). In case B 1*b* the solutions of (1·1) take on different values at z_0^+ and z_0^-, and thus we have effectively two distinct turning points of (1·1), and these are separated symmetrically by the regular singularity $z = 0$. Now the asymptotic character of the solutions of (1·1) depends upon the singularities and turning points of (1·1). The Airy equation

$$\frac{d^2 W}{d\zeta^2} = u^2 \zeta W, \tag{1·3}$$

in terms of the solutions of which the expansions in I were developed, has only one turning point at $\zeta = 0$. This indicates why the Airy-type expansions derived for both cases B 1*b* and B 4 are valid at only one of the two turning points that occur in each case.

In the present paper are derived expansions for case B 1*b* which are valid at $z = z_0^\pm$ and $z = 0$. These expansions are, however, in terms of Bessel functions of large order, and to obtain these expansions we compare (1·1) with the equation

$$\frac{d^2 y}{dt^2} = \left\{u^2\left(1 + \frac{\alpha^2}{t^2}\right) - \frac{1}{4t^2}\right\} y, \tag{1·4}$$

two solutions of which are $t^{\frac{1}{2}} I_m(ut)$, $t^{\frac{1}{2}} K_m(ut)$ where $\alpha = m/u$. We show that α depends upon the behaviour of $p_1(z)$ at $z = 0$; α is kept fixed. The equation (1·4) is clearly an example of case B 4, and as in case B 1*b* the two turning points $t = \pm i\alpha$ are separated symmetrically by the regular singularity $t = 0$. In I, § 5, it was shown that the Legendre functions, as defined by Hobson, are a particular example of case B 1*b*, and the theory given in this paper will be applied later to obtain asymptotic expansions for Legendre functions of large positive degree and order.

It is clear from the above discussion that the asymptotic expansions developed in this paper are similar in character to the earlier expansions obtained by Olver (1954*a*, 1956) for solutions of certain linear second order differential equations. These earlier expansions have been classified according to the type of functions in terms of which the expansions are developed. In his two papers, Olver has considered certain cases described as cases A, B, C and D, and following this classification the theorem stated in § 5 of this paper, giving expansions in terms of Bessel functions of large order, will be called theorem E. It will be shown in § 2 that in case B 1*b*, the equation (1·1) can be transformed into the equation (2·4) which will therefore be described as case E of the equation (1·1*a*).

Asymptotic expansions in terms of Bessel functions of fixed order have been given before by Olver (1956), and certain expansions with Bessel functions of large order have been given by Cherry (1950, § 5·4). These are discussed in detail in § 7. Although the expansions given here were developed independently of Olver's, the two proofs for the expansions are similar. For this reason the theory of this present paper has been considerably rewritten and extended, and much detail has been suppressed by frequent reference to similar, though not identical, sections of Olver's theory.

The arrangement of this paper is as follows. In § 2 a transformation is applied to bring (1·1) into a form suitable for comparison with (1·4). The form of the asymptotic series and relevant properties of the Bessel functions are given in §§ 3 and 4. Theorem E, which states the existence of the asymptotic expansions in terms of Bessel functions, is given in § 5, and the outline of the proof of theorem E is given in § 6. Finally, the expansions given by Olver and Cherry are discussed in § 7.

2. Preliminary transformation to standard form

The Bessel-type expansions for solutions of (1·1) are not derived by an examination of (1·1) as it stands. Instead, we first transform (1·1) into an equation that is asymptotic to (1·4) for large values of u. Now, under the transformation $t \equiv t(z)$, $Y = wz'^{-\frac{1}{2}}$, where $z' \equiv dz/dt$, equation (1·1) becomes

$$\frac{d^2Y}{dt^2} = \{u^2z^{-2}(z_0-z)\,p_1(z)\,z'^2 + g_1(t)\}\,Y, \tag{2·1}$$

where
$$g_1(t) = z'^2z^{-2}q_1(z) + \frac{3z''^2-2z'z'''}{4z'^2} = z'^2z^{-2}q_1(z) - \tfrac{1}{2}\{z,t\}, \tag{2·2}$$

where $\{z, t\}$ is known as the Schwarzian derivative of z with respect to t. We then compare the coefficients of u^2Y in (2·1) and u^2y in (1·4), and write

$$\int_{z_0+}^{z} \{z^{-2}(z_0-z)\,p_1(z)\}^{\frac{1}{2}}\,dz = -\rho = \int_{-i\alpha}^{t} \left(1+\frac{\alpha^2}{t^2}\right)^{\frac{1}{2}} dt. \tag{2·3}$$

The choice of the sign before the last integral in (2·3) is explained below. We then have from (2·1) and (2·2)

$$\frac{d^2Y}{dt^2} = \left\{u^2\left(1+\frac{\alpha^2}{t^2}\right) - \frac{1}{4t^2} + g(t)\right\} Y, \tag{2·4}$$

where
$$g(t) = z'^2 z^{-2} q_1(z) + \frac{1}{4t^2} - \tfrac{1}{2}\{z, t\}. \tag{2·5}$$

The point $z = z_0^+$ corresponds to $\rho = 0$, $t = -i\alpha$, and as $z \to 0$, $\rho \to +\infty$ and $t \to 0$. The function $g(t)$ in (2·5) is not regular at $t = 0$ as it stands. However, we can change the value of the parameter b in (1·1) by replacing u^2 in (1·1) by $u^2 + c$, where c is fixed as $u \to \infty$. Then $q_1(z)$ is replaced by $q_1(z) - c(z_0 - z)\,p_1(z)$ and b by $b - cz_0 a^2$. It is now shown that by choosing b and α appropriately we can make $g(t)$ regular at $t = 0$. From (2·3) and (1·1) we deduce that as $z \to 0$, $t \to 0$ and $z \sim Ct^{\alpha/a}(1 + Ct^r)$, where $r = \min(\alpha/a, 2)$ and C is a generic constant. Substitution in (2·5) gives

$$g(t) = \tfrac{1}{4}\alpha^2(4b+1)\,a^{-2}t^{-2} + Ct^{-2+\alpha/a}(1 + Ct^r) + O(t^{r-2}), \tag{2·6}$$

providing $\alpha \neq a$. If now we set $b = -\frac{1}{4}$—the same condition as required in (3·6) in I—and $\alpha = 2a$, we deduce that $g(t)$ is regular at $z = 0$ $(t = 0)$ and at $z = z_0^+$ $(t = -i\alpha)$. It is shown below that the point $z = z_0^-$ is a regular point of (2·3); if we had chosen $\alpha = a$ this would not be so.

If now we set $\rho = \frac{2}{3}\zeta^{\frac{3}{2}}$ and let $W = \dot{z}^{-\frac{1}{2}}w$, $\dot{z} \equiv dz/d\zeta$ in (1·1), we obtain the z-ρ-ζ transformation used in I to derive the Airy-type asymptotic solutions of (1·1) valid at $z = z_0^+$ and $z = 0$, but not valid at $z = z_0^-$. We conclude that

$$\frac{d^2W}{d\zeta^2} = \{u^2\zeta + f(\zeta)\}\,W, \tag{2·7}$$

where
$$f(\zeta) = \dot{z}^2 z^{-2} q_1(\zeta) - \tfrac{1}{2}\{z, \zeta\} = \dot{z}^2 z^{-2} q_1(z) - \tfrac{1}{2}\dot{t}^2\{z, t\} - \tfrac{1}{2}\{t, \zeta\}, \tag{2·8}$$

where $\dot{t} \equiv dt/d\zeta$, from (2·3) and (2·5). The $\dot{z}$-ρ-ζ transformation is discussed fully in I, § 3. The t-ρ-ζ transformation has also been given before by Olver (1954*b*, pp. 335–337), although in a slightly different form. His variables n, z, ρ and ζ are equivalent to the variables m, $e^{\frac{1}{2}\pi i}\,t/\alpha$, ρ/α, ζ/α of this paper. It follows that

$$\tfrac{2}{3}\zeta^{\frac{3}{2}} = \rho = \alpha \ln \frac{\alpha + \sqrt{(t^2+\alpha^2)}}{it} - \sqrt{(t^2+\alpha^2)}. \tag{2·9}$$

The point $z = z_0^-$ becomes (from I, (4·1)) $\rho = -i\alpha\pi$, $t = i\alpha$ and the three points $z = 0$, $z = z_0^{\pm}$ are regular points of the z-t transformation, and $g(t)$ is regular at these three points.

As in I, we denote by A_ρ, A_t, ..., $\mathbf{D}_\rho$, $\mathbf{D}_t$, ... the points and domains in the ρ, t, ... planes corresponding to A and $\mathbf{D}_z$ in the z plane. Then we deduce that $\mathbf{D}_t$ lies wholly within $\mathscr{R}t > 0$ (see figures 1, 2 and 3). We have chosen the signs in (2·3) so that $\mathbf{D}_t$ would lie in this half-plane.

The domain $\mathbf{T}_z$ surrounding $z = 0$, obtained in I, § 3, is transformed into a domain $\mathbf{T}_t$ surrounding $t = 0$, the boundary of which leaves B_t and B_{1t} at angles of $\frac{1}{3}\pi$ with $\mathscr{R}t = 0$; $\mathbf{T}_t$ is thus one-half of the domain $\mathbf{K}$ of Watson (1944, pp. 270, 559) and Olver (1954*b*, p. 336), rotated through an angle of $-\frac{1}{2}\pi$, the size of $\mathbf{T}_t$ being that of $\mathbf{K}$ magnified by the factor α. We shall denote $\mathbf{T}_t$ by $\mathfrak{K}'$ to emphasize its relation to $\mathbf{K}$, and we denote by $\mathfrak{K}$ the whole domain $\alpha\mathbf{K}$ rotated through an angle of $-\frac{1}{2}\pi$, where $\alpha\mathbf{K}$ denotes the domain $\mathbf{K}$ magnified by the factor α. The point T_t has affix $\alpha \times 0{\cdot}66274\ldots$ (Olver 1954*b*, p. 335). As $z \to 0$, we conclude from (1·1), (2·3) and (2·9) that

$$\frac{z}{z_0}\exp\left(-\frac{a_1}{\alpha}\right) \sim \exp\left(-\frac{\rho}{\alpha}\right) \sim \frac{\mathrm{ie}}{2\alpha}t, \tag{2·10}$$

where $\alpha = 2a$, $a_1 = \int_0^{z_0}\left[\{p(z)\}^{\frac{1}{2}} - \frac{a}{z}\right]\mathrm{d}z$. It also follows that as $|t| \to \infty$ then $|\rho| \to \infty$ with $-\frac{1}{2}\pi > \arg\rho > -\frac{3}{2}\pi$, and

$$t \sim -\rho - \tfrac{1}{2}\mathrm{i}\alpha\pi. \tag{2·11}$$

It will be proved that the asymptotic expansion of the solution $w_1(z)$ of (1·1) which is bounded at $z = 0$ is in terms of $I_m(ut)$. It is exponentially small in $\mathbf{T}_z$ as $u \to \infty$.

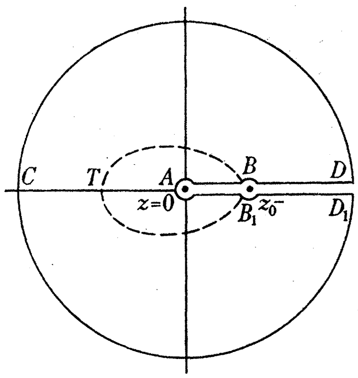

FIGURE 1
Case B 1*b*, *z* plane.

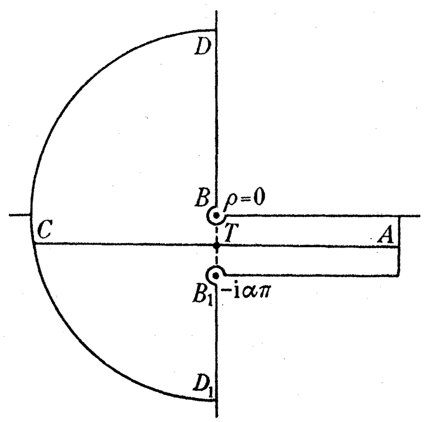

FIGURE 2
Case B 1*b*, *ρ* plane.

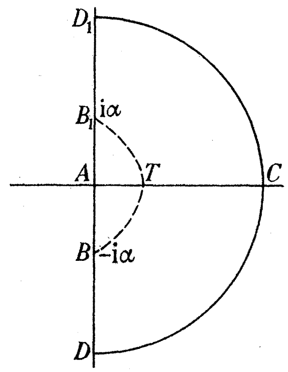

FIGURE 3
Case B 1*b*, *t* plane.

The solutions of (1·1) can be continued analytically across the cuts A_zD_z, A_zD_{1z}, and we can take a Riemann surface $\mathbf{D}_z^*$, consisting of $\mathbf{D}_z$ and two other domains identical with $\mathbf{D}_z$, reached by crossing A_zD_z and A_zD_{1z} once in the negative and positive directions respectively. $\mathbf{D}_t^*$ is a domain lying in $|\arg t| \leqslant \frac{3}{2}\pi$ in which $t = \pm\mathrm{i}\alpha$ are interior points. In this domain it can be shown that $g(t)$ is an even function of t. From (2·5) and (2·8) we can find $g(t)$ in terms of $f(\zeta)$, since

$$g(t) = \left(\frac{\mathrm{d}\zeta}{\mathrm{d}t}\right)^2[f(\zeta) + \tfrac{1}{2}\{t, \zeta\}] + \frac{1}{4t^2}.$$

On some reduction this gives

$$g(t) = \frac{t^2+\alpha^2}{t^2\zeta}\left\{f(\zeta) - \frac{5}{16\zeta^2}\right\} + \frac{t^2-4\alpha^2}{4(t^2+\alpha^2)^2} \tag{2·12}$$

(compare Olver 1954*b*, (4·12)). This completes the preliminary discussion on (1·1).

3. Form of the asymptotic series

We now restrict consideration to the equation (2·4) where α is fixed as $u \to \infty$ and take $g(t)$ to be a regular even function of t in an open simply-connected domain $\mathbf{D}$ in which the points $t = 0$, $t = \pm\mathrm{i}\alpha$ are interior points.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

A formal series solution of the equation

$$\frac{d^2Y}{dt^2} = \left\{u^2\left(1+\frac{\alpha^2}{t^2}\right) - \frac{1}{4t^2} + g(t)\right\} Y, \tag{3·1}$$

can be taken in the form

$$Y(t) = t^{\frac{1}{2}} \mathscr{Z}_m(ut) \sum_{s=0}^{\infty} \frac{A_s(t)}{u^{2s}} + \frac{t^{\frac{1}{2}} \mathscr{Z}'_m(ut)}{u} \sum_{s=0}^{\infty} \frac{B_s(t)}{u^{2s}}, \tag{3·2}$$

where $\mathscr{Z}_m(ut) = cI_m(ut) + dK_m(ut)$, c and d being constants and I_m and K_m being the modified Bessel functions. In (3·2) $\mathscr{Z}'_m(ut) = \mathrm{d}\mathscr{Z}_m(ut)/\mathrm{d}(ut)$, and $A_s(t)$, $B_s(t)$ are analytic functions of t independent of u. If we differentiate the series (3·2) twice and substitute the resulting series and (3·2) into (3·1), and equate powers of u^{-1}, we obtain

$$A''_s + \frac{1}{t} A'_s - gA_s + 2\left(1+\frac{\alpha^2}{t^2}\right) B'_s - \frac{2\alpha^2}{t^3} B_s = 0, \tag{3·3}$$

$$2A'_{s+1} + B''_s - \frac{1}{t} B'_s + \frac{1}{t^2} B_s - gB_s = 0, \tag{3·4}$$

where $A'_s(t) = \mathrm{d}A_s(t)/\mathrm{d}t$. These equations can be integrated to give

$$B_s(t) = \frac{1}{2}\left(1+\frac{\alpha^2}{t^2}\right)^{-\frac{1}{2}} \int_{-\mathrm{i}\alpha}^{t} \left\{g(v)\, A_s(v) - \frac{1}{v} A'_s(v) - A''_s(v)\right\} \left(1+\frac{\alpha^2}{v^2}\right)^{-\frac{1}{2}} \mathrm{d}v, \tag{3·5}$$

$$A_{s+1}(t) = -\tfrac{1}{2} B'_s(t) + \frac{1}{2t} B_s(t) + \frac{1}{2} \int^{t} g(t)\, A_s(t)\, \mathrm{d}t. \tag{3·6}$$

The lower limit in (3·5) is taken to be $t = -\mathrm{i}\alpha$ so that $B_s(t)$ is regular at this point. The integration constant in (3·6) is arbitrary and the arguments of the square roots taken so that $\arg(t-\mathrm{i}\alpha) = \arg(t+\mathrm{i}\alpha) = \arg t = -\frac{1}{2}\pi$ for $t = -\mathrm{i}k$, $k > \alpha > 0$ and k real. Setting

$$A_0(t) = 1, \tag{3·7}$$

we see that $A_s(t)$ is a regular even function of t in **D**, and $B_s(t)$ is a regular odd function of t in **D**, and they are regular at the point $t = \mathrm{i}\alpha$. This follows since $g(t)$ is an even function of t.

In theorem E of § 5, we specify the conditions under which the formal series (3·2) represents the asymptotic solutions of (3·1) in the particular cases $\mathscr{Z}_m(ut) = I_m(ut)$ and $\mathscr{Z}_m(ut) = K_m(ut)$. These two particular Bessel functions are chosen since $I_m(t)$ is bounded for $|t|$ small and $K_m(t)$ is bounded for $|t|$ large and $\mathscr{R}t > 0$. The expansions involving $I_m(ut)$ are proved for all values of $\arg t$, those involving $K_m(ut)$ for $|\arg t| < \frac{3}{2}\pi$. The expansions obtained for the Legendre functions later only require consideration of the half-plane $|\arg t| \leqslant \frac{1}{2}\pi$, but the wider range of $\arg t$ will be considered for completeness.

4. Relevant properties of the Bessel functions

Before stating theorem E we note certain relevant properties of the Bessel functions. We have, as in § 3,

$$\mathscr{Z}_m(t) = cI_m(t) + dK_m(t), \tag{4·1}$$

where c and d are constants. Let $\mathbf{S}_{1t}$, $\mathbf{S}_{2t}$ and $\mathbf{S}_{3t}$ be the regions $|\arg t| \leqslant \frac{1}{2}\pi$, $|\arg t+\pi| \leqslant \frac{1}{2}\pi$

and $|\arg t-\pi|\leqslant\frac{1}{2}\pi$ respectively. Then $K_m(ut)$ is exponentially small in $\mathbf{S}_{1t}$ as $|ut|\to\infty$. We have from Watson (1944, chapter 3)

$$I_m(t)=\sum_{s=0}^{\infty}\frac{(\frac{1}{2}t)^{m+2s}}{s!\,\Gamma(m+s+1)}=\mathrm{e}^{-\frac{1}{2}m\pi\mathrm{i}}J_m(t\,\mathrm{e}^{\frac{1}{2}\pi\mathrm{i}}), \tag{4·2}$$

$$K_m(t)=\tfrac{1}{2}\pi\operatorname{cosec}m\pi\{I_{-m}(t)-I_m(t)\}=\tfrac{1}{2}\mathrm{i}\pi\,\mathrm{e}^{\frac{1}{2}m\pi\mathrm{i}}H_m^{(1)}(\mathrm{i}t), \tag{4·3}$$

$$\left.\begin{aligned}I_m(t\,\mathrm{e}^{r\pi\mathrm{i}})&=\mathrm{e}^{rm\pi\mathrm{i}}I_m(t),\quad r\text{ an integer},\\ K_m(t\,\mathrm{e}^{r\pi\mathrm{i}})&=\mathrm{e}^{-rm\pi\mathrm{i}}K_m(t)-\mathrm{i}\pi\sin rm\pi\operatorname{cosec}m\pi I_m(t),\quad r\text{ an integer.}\end{aligned}\right\} \tag{4·4}$$

We also have the Wronskian

$$\{t^{\frac{1}{2}}I_m(ut),t^{\frac{1}{2}}K_m(ut)\}=u. \tag{4·5}$$

We assume that m and u are large positive numbers, $m=\alpha u$, where α is fixed. Olver (1954*b*, §2) has given the expansions, valid for $|\arg t|\leqslant\frac{1}{2}\pi-\theta$ where $\theta>0$,

$$K_m(ut)\sim\frac{\surd\pi\,\mathrm{e}^{-u\xi}}{\surd(2u)\,(t^2+\alpha^2)^{\frac{1}{4}}}\sum_{s=0}^{\infty}\frac{U_s(\eta)}{(-m)^s},\quad K_m'(ut)\sim-\frac{\surd\pi\,(t^2+\alpha^2)^{\frac{1}{4}}}{\surd(2u)\,t}\mathrm{e}^{-u\xi}\sum_{s=0}^{\infty}\frac{V_s(\eta)}{(-m)^s}, \tag{4·6}$$

$$I_m(ut)\sim\frac{\mathrm{e}^{u\xi}}{\surd(2\pi u)\,(t^2+\alpha^2)^{\frac{1}{4}}}\sum_{s=0}^{\infty}\frac{U_s(\eta)}{m^s},\quad I_m'(ut)\sim\frac{(t^2+\alpha^2)^{\frac{1}{4}}}{\surd(2\pi u)\,t}\mathrm{e}^{u\xi}\sum_{s=0}^{\infty}\frac{V_s(\eta)}{m^s}, \tag{4·7}$$

where

$$\xi=\alpha\eta=-\rho-\tfrac{1}{2}\mathrm{i}\alpha\pi=\alpha\ln\frac{t}{\alpha+\surd(t^2+\alpha^2)}+\surd(t^2+\alpha^2). \tag{4·8}$$

The functions U_s, V_s are given by Olver (1954*b*, (2·19), (2·22)) and the variables z, ζ of Olver (1954*b*, §2) correspond to the variables t/α, $\xi/\alpha=\eta$ of (4·8). From (2·10) and (2·11) we have

$$\text{as}\quad|t|\to\infty,\ \xi\sim t\quad\text{and as}\quad|t|\to0,\ 2\alpha\exp\{(\xi/\alpha)-1\}\sim-t. \tag{4·9}$$

The t-ξ transformation is discussed by Olver and the following remarks will be sufficient for our purpose. $\mathbf{S}_{1t}$ corresponds to a region $\mathbf{S}_{1\xi}$ consisting of $\mathscr{R}\xi\geqslant0$, and $\mathscr{R}\xi\leqslant0$, $|\mathscr{I}\xi|\leqslant\frac{1}{2}\pi\alpha$, shown in figure 4. Let $\mathscr{C}_t(C)$ denote the curve in the t plane which corresponds to the line $\mathscr{R}\xi=C$, where C is a constant, in the ξ plane. Then $\mathscr{C}_t(C)$ is a level curve of $\exp(-\xi)$; that is, it is a curve along which $|\exp(-\xi)|=$ constant. Then $\mathscr{C}_t(0)$ is the boundary of the domain $\mathfrak{K}'$ and the points of the imaginary t-axis which lie outside $\mathfrak{K}'$. For $C>0$, the curves $\mathscr{C}_t(C)$ are asymptotically parallel to the imaginary t axis for large $|t|$ but near the real t axis they are displaced to the right and curl around $\mathfrak{K}'$. For $C<0$, $\mathscr{C}_t(C)$ are finite curves within $\mathfrak{K}'$ and as $C\to-\infty$ they become semicircles around $t=0$. For t in $\mathscr{R}t<0$, $\mathscr{C}_t(C)$ are the reflexion in the imaginary t axis of the curves $\mathscr{C}_t(C)$ in $\mathbf{S}_{1t}$. In the t surface $\mathbf{S}_{1t}+\mathbf{S}_{2t}+\mathbf{S}_{3t}$, $\mathscr{C}_t(C)$ for $C<0$ are finite curves which wind around $t=0$ and have their end-points on the lines $\arg t=\pm\frac{3}{2}\pi$. The curves $\mathscr{C}_t(C)$ are sketched in figure 5 as broken lines, except for the boundary of $\mathfrak{K}'$.

The expansions (4·6) and (4·7) give useful bounds for I_m, K_m in the range $|\arg t|\leqslant\frac{1}{2}\pi-\theta$, $\theta>0$. Outside this range use may be made of the Airy-type expansions of Olver (1954*b*, (4·24), (4·25)) using (4·2), (4·3) together with the continuation formulae (4·4). We then deduce that

$$\left.\begin{aligned}|I_m(ut)|&<S_u(t)=A(1+|t|^{\frac{1}{2}})^{-1}(1+u^{\frac{1}{2}})^{-1}\exp\{u\xi^{(1)}(t)\},\\ |I_m'(ut)|&<S_u'(t)=|t|^{-1}(1+|t|^{\frac{1}{2}})^2S_u(t),\end{aligned}\right\} \tag{4·10}$$

$$\left.\begin{aligned}|K_m(ut)|&<T_u(t)=A(1+|t|^{\frac{1}{2}})^{-1}(1+u^{\frac{1}{2}})^{-1}\exp\{-u\xi^{(2)}(t)\},\\ |K_m'(ut)|&<T_u'(t)=|t|^{-1}(1+|t|^{\frac{1}{2}})^2T_u(t),\end{aligned}\right\} \tag{4·11}$$

where A is a generic constant and $\xi^{(1)}(t)$, $\xi^{(2)}(t)$ are given by

(i) $\xi^{(1)}(t) = \xi^{(2)}(t) = \xi(t)$ for t in $\mathbf{S}_{1t}$,

(ii) for t not in $\mathfrak{R}$, t in $\mathbf{S}_{2t}$ or $\mathbf{S}_{3t}$

$$\xi^{(1)}(t) = -\xi^{(2)}(t) = \xi(t\,\mathrm{e}^{\pm\pi\mathrm{i}}),$$

where the upper or lower signs are taken according as t lies in $\mathbf{S}_{2t}$ or in $\mathbf{S}_{3t}$.

(iii) t lies in $\mathfrak{R}$ and in $\mathbf{S}_{2t}$ or $\mathbf{S}_{3t}$, $\xi^{(1)}(t) = \xi^{(2)}(t) = \xi(t\,\mathrm{e}^{\pm\pi\mathrm{i}})$ with the same convention with respect to signs.

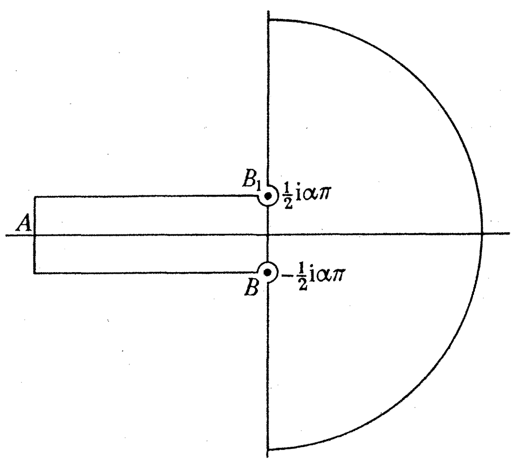

FIGURE 4

Bessel expansions, ξ plane.

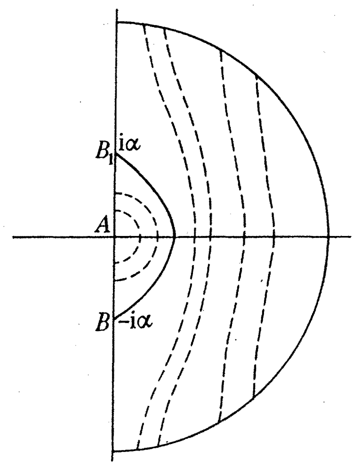

FIGURE 5

Bessel expansions, t plane; curves $\mathscr{C}_t(C)$.

5. The asymptotic series: statement of theorem E

We suppose that $g(t)$ is a regular even function of t in an open simply-connected domain $\mathbf{D}$ in which $t = 0$, $t = \pm\mathrm{i}\alpha$ are interior points; the boundary of $\mathbf{D}$ consists of a finite number of straight lines. If $\mathbf{D}$ is unbounded it is supposed that

$$g(t) = O(|t|^{-1-\sigma}) \quad (\sigma > 0), \tag{5·1}$$

as $|t| \to \infty$ in $\mathbf{D}$, uniformly with respect to $\arg t$, where σ is a constant. $\mathbf{D}'$ denotes any simply-connected domain lying wholly within $\mathbf{D}$ with $t = 0$, $t = \pm\mathrm{i}\alpha$ as interior points, and having a boundary which does not intersect the boundaries of $\mathbf{D}$, and which also consists of a finite number of straight lines. In figure 6 the boundary of $\mathbf{D}$ is given by firm lines, and that of $\mathbf{D}'$ by broken lines.

We define a further domain $\mathbf{D}_0$ to consist of those points t of $\mathbf{D}'$ which can be joined to the origin by a curve which does not cross either the imaginary t axis or the level curve $\mathscr{C}_t(C)$ through t. Finally, we define a domain $\mathbf{D}_1$ in relation to a point a_1 which lies in $\mathbf{S}_{1t}$ and $\mathbf{D}'$ and which may be at infinity in $\mathbf{D}'$. $\mathbf{D}_1$ consists of all points t of $\mathbf{D}'$ for which $|\arg t| \leqslant \frac{3}{2}\pi$ and which can be joined to a_1 by a curve $\mathscr{A}_t$ which lies in $\mathbf{D}'$ and satisfies the following conditions:

(i) if t lies outside $\mathfrak{R}$ then $\mathscr{A}_t$ must lie outside $\mathfrak{R}$ and to the right of the level curve $\mathscr{C}_t(C)$ through t;

(ii) if t lies inside $\mathfrak{R}$ then $\mathscr{A}_t$ must lie outside the (closed) level curve $\mathscr{C}_t(C)$ through t. If a point with affix v lies on $\mathscr{A}_t$, then

$$|\exp\{-\xi^{(2)}(t)\}| \geqslant |\exp\{-\xi^{(2)}(v)\}| \geqslant |\exp\{-\xi^{(2)}(a_1)\}|.$$

An example of a domain $\mathbf{D}_0$ is given in figure 7, and that part of $\mathbf{D}_1$ for which $-\frac{1}{2}\pi \leqslant \arg t \leqslant \frac{3}{2}\pi$, in figure 8.

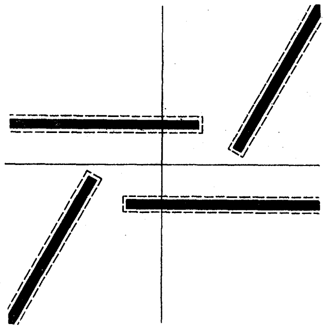

FIGURE 6
Domains $\mathbf{D}$, $\mathbf{D}'$.

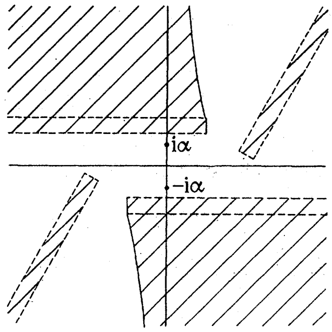

FIGURE 7
Domain $\mathbf{D}_0$.

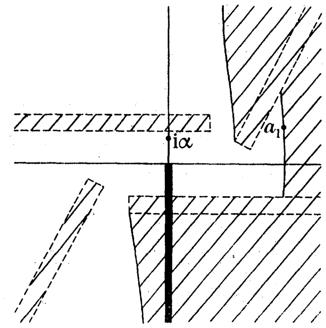

FIGURE 8
Domain $\mathbf{D}_1$, $-\frac{1}{2}\pi \leqslant \arg t \leqslant \frac{3}{2}\pi$.

THEOREM E. *Let the functions* $A_s(t)$, $B_s(t)$ $(s = 0, 1, 2, \ldots)$ *be defined by the relations* (3·5), (3·6) and (3·7). Let $C_s(t)$, $D_s(t)$ *be defined by the relations*

$$C_s(t) = tA_s'(t) + \tfrac{1}{2}A_s(t) + t\left(1 + \frac{\alpha^2}{t^2}\right)B_s(t),$$

$$D_s(t) = A_s(t) + B_{s-1}'(t) - \frac{1}{2t}B_{s-1}(t).$$

Then the equation
$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} = \left\{u^2\left(1 + \frac{\alpha^2}{t^2}\right) - \frac{1}{4t^2} + g(t)\right\}y, \tag{5·2}$$

where $g(t)$ *satisfies the conditions given above and* α *is real, has solutions* $Y_0(t)$, $Y_1(t)$ *such that*

(i) If t lies in $\mathbf{D}_0$

$$Y_0(t) = t^{\frac{1}{2}}I_m(ut)\left\{\sum_{s=0}^{N-1}\frac{A_s(t)}{u^{2s}} + O\left(\frac{1}{u^{2N}}\right)\right\} + \frac{t^{\frac{1}{2}}I_m'(ut)}{u}\left\{\sum_{s=0}^{N-1}\frac{B_s(t)}{u^{2s}} + \frac{t}{1+|t|}O\left(\frac{1}{u^{2N}}\right)\right\}, \tag{5·3}$$

$$Y_0'(t) = t^{-\frac{1}{2}}I_m(ut)\left\{\sum_{s=0}^{N-1}\frac{C_s(t)}{u^{2s}} + (1+|t|)\,O\left(\frac{1}{u^{2N}}\right)\right\} + ut^{\frac{1}{2}}I_m'(ut)\left\{\sum_{s=0}^{N-1}\frac{D_s(t)}{u^{2s}} + O\left(\frac{1}{u^{2N}}\right)\right\}, \tag{5·4}$$

as $u \to \infty$, *where the* O'*s are uniform with respect to* t.

(ii) If t lies in $\mathbf{D}_1$

$$Y_1(t) = t^{\frac{1}{2}}K_m(ut)\left\{\sum_{s=0}^{N-1}\frac{A_s(t)}{u^{2s}} + O\left(\frac{1}{u^{2N}}\right)\right\} + \frac{t^{\frac{1}{2}}K_m'(ut)}{u}\left\{\sum_{s=0}^{N-1}\frac{B_s(t)}{u^{2s}} + \frac{t}{1+|t|}O\left(\frac{1}{u^{2N}}\right)\right\}, \tag{5·5}$$

$$Y_1'(t) = t^{-\frac{1}{2}}K_m(ut)\left\{\sum_{s=0}^{N-1}\frac{C_s(t)}{u^{2s}} + (1+|t|)\,O\left(\frac{1}{u^{2N}}\right)\right\} + ut^{\frac{1}{2}}K_m'(ut)\left\{\sum_{s=0}^{N-1}\frac{D_s(t)}{u^{2s}} + O\left(\frac{1}{u^{2N}}\right)\right\}, \tag{5·6}$$

as $u \to \infty$, *where the* O'*s are uniform with respect to* t.

In (5·3) to (5·6) N is an arbitrary positive integer, and $Y_0(t)$, $Y_1(t)$ are independent of N. Thus we can write the series (5·3) as

$$Y_0(t) \sim t^{\frac{1}{2}}I_m(ut)\sum_{s=0}^{\infty}\frac{A_s(t)}{u^{2s}} + \frac{t^{\frac{1}{2}}I_m'(ut)}{u}\sum_{s=0}^{\infty}\frac{B_s(t)}{u^{2s}}, \tag{5·7}$$

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

and there are similar infinite expansions corresponding to each of (5·4) to (5·6). Compare with (3·2) and (4·1). We note that $I_m'(ut) = \mathrm{d}\{I_m(ut)\}/\mathrm{d}(ut)$.

The equation (1·1). The function $I_m(ut)$ is exponentially small as $|t| \to 0$ in the domain $\mathfrak{K}$ surrounding $t = 0$; $\mathfrak{K}'$ corresponds to the domain $\mathbf{T}_z$ surrounding $z = 0$ in the z plane of equation (1·1), and the solution $w_1(z)$ of (1·1), which is bounded at $z = 0$, will therefore have an expansion in terms of $I_m(ut)$.

In § 1 it was shown that the z-ρ-t transformation (2·3), used to bring the equation (1·1) into the form (2·4) suitable for the application of theorem E, is related to the z-ρ-ζ transformation of I, (2·5), used to obtain Airy-type expansions for solutions of (1·1); it is known that there exist Airy-type expansions for solutions of (2·7) in any domain $\mathbf{D}_\zeta$ in which $f(\zeta)$ is regular and for which $f(\zeta) = O(|\zeta|^{-\frac{1}{2}-\mu})$ $(\mu > 0)$, as $|\zeta| \to \infty$ in $\mathbf{D}_\zeta$. This condition on $f(\zeta)$ is related to the condition (5·1) on $g(t)$ as $|t| \to \infty$ in $\mathbf{D}_t$. For, as $|\zeta| \to \infty$ in $|\arg(-\zeta)| < \frac{2}{3}\pi$, then $|t| \to \infty$, and from (2·5), (2·8), (2·9), (2·11) and (2·12) we deduce that $g(t) = O(|t|^{-1-\sigma})$, $\sigma = \frac{2}{3}\mu > 0$, as $|t| \to \infty$.

6. Proof of theorem E

There are two stages to the proof of theorem E; first, several properties of the functions $A_s(t)$, $B_s(t)$ are derived and stated in lemmas 1, 2 and 3 below, and then the theorem itself is proved using these properties. No proofs are given here, but relevant comments are given below and the theory is similar to that of Olver (1956) throughout.

We first define the constants d, R' and the domains $\mathbf{D}_\delta$, $0 < \delta \leqslant d$, in relation to $\mathbf{D}$, $\mathbf{D}'$ in the same way as is done in Olver (1954*a*, § 8), except that here the points $t = \pm i\alpha$, $t = 0$ are interior points of $\mathbf{D}$, $\mathbf{D}'$, $\mathbf{D}_\delta$. Let α' be a positive number such that $\alpha' > \alpha$, and let

$$R = \max(R' + d, 3\alpha').$$

If t lies in $\mathbf{D}_\delta$ and $|t| > R$, then t lies in one of a number of unbounded subdomains, $\mathbf{E}_\delta^{(1)}$, $\mathbf{E}_\delta^{(2)}$, ..., say, denoted typically by $\mathbf{E}_\delta$. The following lemmas are proved with the restriction that $A_s(c) = 0$, for some point c in $\mathbf{D}_d$. This restriction can be relaxed in a manner similar to that given in Olver (1954*a*, § 11). Let $\kappa = \min(\sigma, 1)$, σ defined in (5·1).

Lemma 1. *If* $|t| \to \infty$ *in* $\mathbf{E}_\delta$, *then*

$$A_s(t) = \alpha_s + O(|t|^{-\kappa}), \quad B_s(t) = \beta_s + O(|t|^{-\kappa}), \quad A_s'(t) = O(|t|^{-1-\kappa}), \tag{6·1}$$

uniformly with respect to $\arg t$ *in* $\mathbf{E}_\delta$. *Here* α_s *and* β_s *are constants.*

Lemma 2. *Let* t *lie in* $\mathbf{D}_\delta$ *and let* $|v| < V$, *where* $V \equiv V(\delta) > 0$ *is an assignable constant independent of* t. *Then the series*

$$A(t, v) = \sum_{s=0}^{\infty} A_s(t) \frac{v^{2s}}{(2s)!}, \quad B(t, v) = \sum_{s=0}^{\infty} B_s(t) \frac{v^{2s}}{(2s)!}, \tag{6·2}$$

converge uniformly with respect to t *and* v.

Lemma 3. *Let* t *lie in* $\mathbf{E}_\delta$, *and let* $|v| < V_0$, *where* V_0 *is assignable, independent of* t *and* $0 < V_0 \leqslant V$. *Then*

$$S_1(t, v) \equiv \frac{\partial}{\partial t} A(t, v) = \sum_{s=0}^{\infty} A_s'(t) \frac{v^{2s}}{(2s)!} = O(|t|^{-1-\kappa}), \tag{6·3}$$

$$S_2(t, v) \equiv \frac{\partial^2}{\partial t \partial v} A(t, v) = \sum_{s=1}^{\infty} A_s'(t) \frac{v^{2s-1}}{(2s-1)!} = O(|t|^{-1-\kappa}), \tag{6·4}$$

as $|t| \to \infty$ *uniformly with respect to* v *and* $\arg t$.

The method of the proof of these lemmas is basically the same as that used by Olver (1956, pp. 81–86) for his lemmas 2, 3 and 4, except that in lemma 2 above the integral in (3·5) is integrated by parts twice whereas in Olver's theory (1956, (10·30)) only one such integration is required.

These lemmas having been established, the proof of theorem E follows in exactly the same way as that for theorem D in Olver (1956, §§ 11–13), except that the inequalities (4·10) and (4·11) are used instead of Olver's inequalities for Bessel functions of fixed order (1956, (9·4), (9·5), (9·10), (9·11)).

7. Previous results

If $f(z)$ is a regular even function of z in a domain $\mathbf{D}_z$ in which $z = 0$ is an interior point, and if μ is a fixed parameter with $\mathscr{R}\mu \geqslant 0$, Olver (1956) has shown that there exist, as $u \to \infty$, asymptotic expansions for solutions of the differential equation

$$\frac{d^2w}{dz^2} = \left\{u^2 + \frac{\mu^2}{z^2} - \frac{1}{4z^2} + f(z)\right\} w, \tag{7·1}$$

uniformly valid with respect to z in subdomains of $\mathbf{D}_z$ including the origin; the expansions are in terms of the Bessel functions $I_\mu(uz)$, $K_\mu(uz)$. The equation (7·1) has no turning points since μ is fixed as $u \to \infty$. It is clear that if the parameter μ is allowed to become large, (7·1) will become an equation of the type (5·2), and the theory of this present paper is relevant.

Cherry (1950, § 5·4) has obtained certain expansions in terms of Bessel functions of large order for solutions of equations of type (1·1). To obtain these expansions he first derives Airy-type *approximations* for the solutions of (1·1) of the form

$$\psi(\zeta, u)\,\mathrm{Ai}\,(u^{\frac{2}{3}}\zeta_m)\,\{1 + O(u^{-2m}\zeta^{-\frac{1}{4}})\}, \quad \zeta_m = \sum_{r=0}^{\infty} u^{-2r}\phi_{r,m}(\zeta), \quad \zeta = \tfrac{3}{2}\rho^{\frac{2}{3}}.$$

Similar Airy-type approximations can also be derived for the Bessel functions $I_m(ut)$, $K_m(ut)$, and by eliminating the Airy functions between these, we can obtain an approximation—and thence an expansion—for the solutions of (1·1). However, since the Airy-type approximation for the Bessel functions are not valid at $t = +\mathrm{i}\alpha$, and are not uniformly valid in the strip $|\mathscr{R}t| < \delta$, $\mathscr{I}t > \mathrm{i}\alpha - \delta$ $(\delta > 0)$, we cannot prove that the Bessel-type approximations and expansions obtained by Cherry's method are uniformly valid at these points. Since we have had as our aim the development of expansions valid uniformly at these points, it has been necessary to develop the theory given in this paper. Cherry's theory, however, extends to cases when u is not necessarily real and his results hold for *all* large u (1950, p. 256). In this paper u is a large real positive parameter.

I wish to thank Mr F. W. J. Olver, Mr F. Ursell and Professor A. Erdélyi for helpful discussions on the subject of the paper. I also wish to acknowledge the generous support of Trinity College, Cambridge, and the University of Cambridge for scholarships and grants, and the U.S. Office of Naval Research for the sponsorship of a research fellowship, during the tenure of which this paper was completed. This present paper has also been produced as Technical Report number 12, Contract Nonr-220(11), Department of Mathematics, California Institute of Technology.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

References

Cherry, T. M. 1950 *Trans. Amer. Math. Soc.* **68**, 224–257.
Olver, F. W. J. 1954*a* *Phil. Trans.* A, **247**, 307–327.
Olver, F. W. J. 1954*b* *Phil. Trans.* A, **247**, 328–368.
Olver, F. W. J. 1956 *Phil. Trans.* A, **249**, 65–97.
Thorne, R. C. 1957 *Proc. Camb. Phil. Soc.* **53**, 382–398 (referred to as I). Tech. Report 11 under contract Nonr-220(11), CIT, Pasadena, California.
Watson, G. N. 1944 *Theory of Bessel functions*. Cambridge University Press.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES